

The Chow group of a Châtelet surface over a number field

Chandan Singh Dalawat

We compute the Chow group of a Châtelet surface over a dyadic field. Combined with the previous work of Bloch, Colliot-Thélène, Coray, Ischebeck, Sansuc, Swinnerton-Dyer, and the author, this allows one to compute the Chow group of any Châtelet surface over any number field.

1. Introduction

2-descent (also called a first descent) on elliptic curves E defined over \mathbf{Q} which have the form $y^2 = (x - c_1)(x - c_2)(x - c_3)$ (where the c_i are distinct), is a classical theme which goes back to Pierre Fermat, with a major contribution by John Tate. If carried out successfully, one ends up computing the finitely generated commutative group $E(\mathbf{Q})$ of rational points on E .

There is a simpler version of 2-descent, applicable to Châtelet surfaces, i.e. smooth proper surfaces X birational to the affine surface defined by

$$(1) \quad y^2 - dz^2 = (x - c_1)(x - c_2)(x - c_3) \quad (d \in \mathbf{Q}^\times).$$

This version has also been pursued by various authors, whose contributions will be recalled in the course of this Note. We intend to carry it out successfully, computing thereby the Chow group of X , the finite \mathbf{F}_2 -space $A_0(X)_0$ of degree-0 0-cycles modulo rational equivalence.

Of the real 2-descent, the one for elliptic curves, Peter Swinnerton-Dyer wrote recently that “the statements of the theory over an arbitrary number field are not very different, except that the analogues of certain explicit results relating to the prime 2 are not known” [8].

The main contribution of this Note is to obtain those explicit results at the prime 2 for 2-descent on Châtelet surfaces; they allow us to compute the Chow group of any Châtelet surface over any number field.

2. Statement of the local results

Let K be a finite extension of the field \mathbf{Q}_p (p prime) of p -adic numbers, \mathfrak{o} the ring of integers of K (i.e. the integral closure of \mathbf{Z}_p in K), \mathfrak{p} the unique maximal ideal of \mathfrak{o} , and $k = \mathfrak{o}/\mathfrak{p}$ the residue field of K . Denote by $v : K^\times \rightarrow \mathbf{Z}$ the surjective valuation of K .

Given $d \in K^\times$ and three distinct numbers $c_1, c_2, c_3 \in K$, we get a smooth affine K -surface

$$(2) \quad y^2 - dz^2 = (x - c_1)(x - c_2)(x - c_3)$$

which is K -birational to \mathbf{P}_2 if $d \in K^{\times 2}$; if $d \notin K^{\times 2}$, it is $K(\sqrt{d})$ -birational to \mathbf{P}_2 , and the group $A_0(X)_0$ is killed by 2.

Let X be any smooth projective K -surface which is K -birational to (2). Thus the Chow group $A_0(X)_0$ of 0-cycles of degree 0 modulo rational equivalence, which we are interested in computing, is trivial if $d \in K^{\times 2}$; we therefore assume that $d \notin K^{\times 2}$.

To fix ideas, we take X to be the surface defined in $\mathbf{P}(\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O})$ (coordinates $y : z : t$) over the projective K -line $\mathbf{P}_{1,K}$ (coordinates $x : x'$) by the equation

$$(3) \quad y^2 - dz^2 = (x - c_1x')(x - c_2x')(x - c_3x')t^2;$$

this model was constructed by Colliot-Thélène and Sansuc in [3]. This choice is immaterial for our purposes, as we are only interested in computing the group $A_0(X)_0$ which is independent of the choice of the projective model, by a result of Colliot-Thélène and Coray [1]. The surface (3) comes equipped with a morphism to \mathbf{P}_1 whose fibres are conics; there are four degenerate fibres, namely the ones above c_1, c_2, c_3 and ∞ .

The change of variables $x_1 = x - c_1$ allows us to write (2) as

$$y^2 - dz^2 = x_1(x_1 - c'_1)(x_1 - c'_2).$$

Upto permuting c'_1, c'_2 , we may suppose that $v(c'_2) \geq v(c'_1)$. Moreover, if $v(c'_2) > v(c'_1)$, the change of variables $x_2 = x_1 - c'_1$ transforms this equation into

$$y^2 - dz^2 = x_2(x_2 - c''_1)(x_2 - c''_2)$$

in which $v(c''_2) = v(c''_1)$. In other words, we can suppose without any loss of generality that we have $c_1 = 0$ and $v(c_2) = v(c_3)$ in (2). Denoting this common valuation by r , we shall henceforth work with

$$(4) \quad y^2 - dz^2 = x(x - e_1)(x - e_2) \quad (r = v(e_i)).$$

With these conventions, let us recall the cases in which the group $A_0(X)_0$ has been computed.

PROPOSITION 1 ([6, Prop. 4.7] for p odd, [7, Prop. 1] for $p = 2$). — Suppose that the extension $K(\sqrt{d})$ is unramified. The group $A_0(X)_0$ is then isomorphic

$$\begin{array}{lll} i) \text{ to} & 0 & \text{if } r \text{ is even and } v(e_1 - e_2) = r, \\ ii) \text{ to} & \mathbf{Z}/2\mathbf{Z} & \text{if } r \text{ is even and } v(e_1 - e_2) > r, \\ iii) \text{ to} & (\mathbf{Z}/2\mathbf{Z})^2 & \text{if } r \text{ is odd.} \end{array}$$

PROPOSITION 2 ([7, Prop. 2]). — Suppose that p is odd and that the extension $L = K(\sqrt{d})$ is ramified. The group $A_0(X)_0$ is then isomorphic

$$\begin{array}{lll} i) \text{ to} & \mathbf{Z}/2\mathbf{Z} & \text{if } e_1/e_2 \equiv 1 \pmod{\mathfrak{p}} \text{ and } e_1 \in N_{L|K}(L^\times), \\ ii) \text{ to} & (\mathbf{Z}/2\mathbf{Z})^2 & \text{if } e_1/e_2 \equiv 1 \pmod{\mathfrak{p}} \text{ and } e_1 \notin N_{L|K}(L^\times), \\ iii) \text{ to} & (\mathbf{Z}/2\mathbf{Z})^2 & \text{if } e_1/e_2 \not\equiv 1 \pmod{\mathfrak{p}}. \end{array}$$

Suppose that $p = 2$ and that the extension $L = K(\sqrt{d})$ is ramified. Let $\chi : K^\times \rightarrow \mathbf{Z}/2\mathbf{Z}$ be the homomorphism whose kernel is the group of norms from L^\times . The restriction of χ to the units \mathfrak{o}^\times is $\neq 0$, because $L|K$ is ramified. Since k^\times is of odd order, $\chi|_{\mathfrak{o}^\times}$ factors via the group U_1 of 1-units, and indeed via U_1/U_{n+1} , but not via U_1/U_n , for a suitable $n > 0$. Here U_n denotes the group of units which are $\equiv 1 \pmod{\mathfrak{p}^n}$.

As the archimedean local fields were treated in [2], the missing ingredient is provided by the next proposition, which is our main result. This proposition not only completes the determination of the Chow group of a Châtelet surface over a local field, but, because of a local-to-global principle which we will recall below, also over number fields.

PROPOSITION 3. — Suppose that $p = 2$ and that the extension $L = K(\sqrt{d})$ is ramified. Suppose that χ factors via U_1/U_{n+1} , but not via U_1/U_n . The group $A_0(X)_0$ is then isomorphic

$$\begin{array}{lll} i) \text{ to} & \mathbf{Z}/2\mathbf{Z} & \text{if } e_1/e_2 \equiv 1 \pmod{\mathfrak{p}^{2n+1}} \text{ and } e_1 \in N_{L|K}(L^\times), \\ ii) \text{ to} & (\mathbf{Z}/2\mathbf{Z})^2 & \text{if } e_1/e_2 \equiv 1 \pmod{\mathfrak{p}^{2n+1}} \text{ and } e_1 \notin N_{L|K}(L^\times), \\ iii) \text{ to} & (\mathbf{Z}/2\mathbf{Z})^2 & \text{if } e_1/e_2 \not\equiv 1 \pmod{\mathfrak{p}^{2n+1}}. \end{array}$$

As we shall see in the course of the proof, these three propositions, as well as the result at the places at ∞ , actually compute $A_0(X)_0$ as a subgroup of $H^1(K, S(\bar{K}))$, where \bar{K} is an algebraic closure of K and S is the K -torus whose group of characters is the $\text{Gal}(\bar{K}|K)$ -module $\text{Pic}(\bar{X})$, with $\bar{X} = X \times_K \bar{K}$. This is crucial for the application to computing the Chow group of a Châtelet surface over a number field.

3. The method of computation

It relies on the work of Colliot-Thélène and Coray [1] and Colliot-Thélène and Sansuc [3], [4]; it has been explained at length in [7]. As this last paper was written in a language which is poorly understood in many parts of the world, ce qui n'est pas sans rapport avec les péripéties de l'auteur, we shall gave a brief summary here.

One can replace X by any smooth proper K -surface K -birational to X [1, Prop. 6.3], which justifies the choice of the particular model X (cf. (3)) of the equation (4) that we have made.

Next, denoting by O the singular point of the fibre at infinity of the conic bundle $f : X \rightarrow \mathbf{P}_1$, the map

$$\gamma : X(K) \rightarrow A_0(X)_0, \quad \gamma(Q) = Q - O$$

is surjective [1, Théorème C]. “The characteristic homomorphism” is a natural injection

$$\varphi : A_0(X)_0 \rightarrow H^1(K, S(\bar{K}))$$

[4, n° IV]. With the identifications

$$\iota : H^1(K, S(\bar{K})) \rightarrow (K^\times / N_{L|K}(L^\times))^2 \rightarrow (\mathbf{Z}/2\mathbf{Z})^2,$$

the composite map $X(K) \rightarrow (\mathbf{Z}/2\mathbf{Z})^2$ is given by

$$(1) \quad (y : z : t; x) \mapsto \begin{cases} (\chi(1), \chi(1)) & \text{if } x = \infty, \\ (\chi(e_1 e_2), \chi(-e_1)) & \text{if } x = 0, \\ (\chi(e_1), \chi(e_1(e_1 - e_2))) & \text{if } x = e_1, \\ (\chi(x), \chi(x - e_1)) & \text{otherwise.} \end{cases}$$

[3, n° IV]. As all the points in the same fibre of the map $f : X(K) \rightarrow \mathbf{P}_1(K)$ are mutually equivalent 0-cycles, what we have to compute is the image of the induced map $[] : f(X(K)) \rightarrow (\mathbf{Z}/2\mathbf{Z})^2$. The subset $f(X(K)) \subset \mathbf{P}_1(K)$ consists of $\infty, 0, e_1, e_2$ and all those $x \in K$, different from $0, e_1, e_2$, for which $\chi(x(x - e_1)(x - e_2)) = 0$. As explained in [7], what we have to compute is the subgroup generated by the image $[f(X(K))]$.

All this is valid for K any extension of \mathbf{Q}_p , and $L = K(\sqrt{d})$ any quadratic extension of K . As explained in [7, Remarque 5], when L is a ramified extension, we can suppose that $v(e_i) = 0$. Further, when $p = 2$, we can assume that $e_i \in U_1$, as every element of k^\times is in $N_{L|K}(L^\times)$.

We assume from now onwards that $p = 2$, that $L|K$ is ramified, and that the e_i are (distinct) 1-units of K .

LEMMA 1. — Suppose that χ factors via $(\mathfrak{o}/\mathfrak{p}^n)^\times \rightarrow \mathbf{Z}/2\mathbf{Z}$. Then, for every $x \in f(X(K))$, we have

$$[x] = \begin{cases} (0, 0) & \text{if } v(x) \leq -n, \\ [0] & \text{if } v(x) \geq n. \end{cases}$$

Proof: For $x \in K$ distinct from 0, e_1, e_2 , we have

$$\chi(x(x - e_1)(x - e_2)) = \chi(x) + \chi(x - e_1) + \chi(x - e_2);$$

if further $x \in f(X(K))$, this sum is 0. If moreover $v(x) \leq -n$, then each term is 0, since $\chi(x - e_i) = \chi(x)$ in this case, so $[x] = (0, 0)$. If however $v(x) \geq n$, we have $\chi(x - e_i) = \chi(-e_i)$ and therefore $\chi(x) = \chi(e_1 e_2)$; accordingly, $[x] = [0]$.

PROPOSITION 4. — Suppose that χ factors via U_1/U_{n+1} but does not factor via U_1/U_n . Then the subgroup generated by $[f(X(K))]$ is :

- i) $\{0\} \times \mathbf{Z}/2\mathbf{Z}$ if $e_1 \equiv e_2 \pmod{\mathfrak{p}^{2n+1}}$ and $\chi(e_1) = 0$,
- ii) $(\mathbf{Z}/2\mathbf{Z})^2$ if $e_1 \equiv e_2 \pmod{\mathfrak{p}^{2n+1}}$ and $\chi(e_1) \neq 0$,
- iii) $(\mathbf{Z}/2\mathbf{Z})^2$ if $e_1 \not\equiv e_2 \pmod{\mathfrak{p}^{2n+1}}$.

We shall identify k^\times with the prime-to-2 torsion subgroup of \mathfrak{o}^\times .

Putting $H = \text{Ker}(\chi)$, notice that $1 + t\bar{\pi}^n \notin H$ for some $t \in k^\times$, since otherwise χ would factor via U_1/U_n . Fix such a $t \in k^\times$; notice that $1 - t\bar{\pi}^n \notin H$, for $1 - t\bar{\pi}^n \equiv 1 + t\bar{\pi}^n \pmod{\mathfrak{p}^{n+1}}$, as $2 \equiv 0 \pmod{\mathfrak{p}^e}$. If $a \in H$, then $a(1 + t\bar{\pi}^n) \notin H$, for their quotient is $1 + t\bar{\pi}^n$. Finally, $H \rightarrow U_1/U_n$ is a surjection, with kernel $H \cap (U_n/U_{n+1})$ of order 2^{f-1} , where $q = 2^f$ is the number of elements in the residue field k .

Put $x = t^{-1}\pi^{-n}$. Observe that $x \in f(X(K))$ and $[x] = (0, 1)$ in all cases : we have $\chi(x) = 0$, and $\chi(x - e_i) = \chi(1 - te_i\pi^n) = 1$, because $1 - te_i\pi^n \equiv 1 - t\pi^n \pmod{\mathfrak{p}^{n+1}}$.

Proof of i) We have $[0] = (0, *)$, $[e_1] = (0, *)$, $[e_2] = (0, *)$; let us show that $(1, *)$ is not (i.e. $(1, 0)$ and $(1, 1)$ are not) in the image $[f(X(K))]$. It is sufficient to show that if $x \in K$ is distinct from 0, e_1, e_2 , if its valuation is between $-n$ and n (cf. lemma 1), and if $\chi(x) = 1$, then $x \notin f(X(K))$. Recalling that $v(e_1 - e_2) > 2n$ and that $v(e_i) = 0$, we deduce that $v(x - e_1) = v(x - e_2)$ is given by

$$\begin{array}{cccccccc} v(x) & -n & \cdots & -1 & 0 & 1 & \cdots & n \\ \hline v(x - e_i) & -n & \cdots & -1 & 0, 1, \dots, n & 0 & \cdots & 0 \end{array}$$

Only the case $v(x) = 0$ needs some explanation : the fact that $\chi(x) = 1$ whereas $\chi(e_i) = 0$ means that $x \not\equiv e_i \pmod{\mathfrak{p}^{n+1}}$, i.e. $v(x - e_i) < n + 1$.

Writing $r = v(x - e_i)$, we have

$$\pi^{-r}(x - e_2) = \pi^{-r}(x - e_1) + \pi^{-r}(e_1 - e_2)$$

with $v(\pi^{-r}(e_1 - e_2)) > n$. So we get $x - e_1 \equiv x - e_2 \pmod{\mathfrak{p}^{n+1}}$, which implies that $\chi(x - e_1) = \chi(x - e_2)$, and hence $x \notin f(X(K))$. This failure contains the seeds of our success in case *iii*) below.

Proof of ii) We have $[e_1] = (1, *)$, as $\chi(e_1) = 1$.

Proof of iii) It remains to show that $(1, *)$ is (i.e. $(1, 0)$ or $(1, 1)$ is) in $[f(X(K))]$, which is clearly the case if $\chi(e_1) = 1$, and also if $\chi(e_2) = 1$. Suppose then that $\chi(e_i) = 0$.

Write $e_1 - e_2 = s_{1,2}u_{1,2}\pi^a$ ($s_{1,2} \in k^\times$, $u_{1,2} \in U_1$, $1 \leq a \leq 2n$) and let us first deal with the case $a = 2n$. Identify the \mathbf{F}_2 -space k with the subgroup U_n/U_{n+1} of U_1/U_{n+1} by the map $s \mapsto 1 + s\pi^n$, and let $h \subset k$ be the codimension-1 subspace such that $\chi(1 + s\pi^n) = 0$ if and only if $s \in h$, i.e. $h = H \cap k$. There is a $t \notin h$ such that $s_{1,2}t^{-1} \notin h$, for, as $t \notin h$ varies, we get $2^f - 2^{f-1} = 2^{f-1}$ distinct elements $s_{1,2}t^{-1}$ of k^\times , of which at most $2^{f-1} - 1$ can belong to h : we are saved by the skin of our teeth. Fix such a t , and put $x = e_1 + t\pi^n$, so that $\chi(x) = 1$ — this is the only place where we are using the fact that $\chi(e_1) = 0$ — and $\chi(x - e_1) = 0$. Also, $v(x - e_2) = n$ and, writing $x - e_2 = (x - e_1) + (e_1 - e_2)$, we get

$$\frac{x - e_2}{\pi^n} \equiv t(1 + s_{1,2}t^{-1}\pi^n) \pmod{\mathfrak{p}^{n+1}},$$

so $\chi(x - e_2) = 1$. Thus, $x \in f(X(K))$ and $[x] = (1, 0)$.

Assume now that the valuation a of $e_1 - e_2$ is $< 2n$ and let $t \in k^\times$ be such that $1 + t\pi^n \notin H$. Put $s = s_{1,2}t^{-1}$ and take $x = e_1 + su\pi^{a-n}$, where $u \in U_1$ is such that $\chi(x) = 1$; this can always be arranged, at the cost of replacing x by $x(1 + t\pi^n)$, i.e. u by the w satisfying $x(1 + t\pi^n) = e_1 + sw\pi^{a-n}$ ($w \in U_1$; in fact, $w = u + tu\pi^n + s^{-1}t\pi^{2n-a}$). We have $v(x - e_i) = a - n$, and

$$\frac{x - e_2}{\pi^{a-n}} = su + s_{1,2}u_{1,2}\pi^n \equiv su(1 + t\pi^n) \pmod{\mathfrak{p}^{n+1}},$$

so $\chi(x - e_2) + \chi(x - e_1) = 1$, i.e. $x \in f(X(K))$. We have $[x] = (1, *)$.

This completes the proof of Prop. 4, and thereby also Prop. 3.

4. Consequence

The main application of Prop. 3 is to the computation of the Chow group of 0-cycles of degree 0 on a Châtelet surface defined over a number

field, an application made possible by the local-to-global principle as conjectured in [4] and proved in [5]. As this process has been clearly explained in [9] and [4], we content ourselves with a practical remark.

Changing notation, let K be a finite extension of \mathbf{Q} , $d \in K^\times$ and let $c_1, c_2, c_3 \in K$ be distinct. Let X be any smooth proper surface K -birational to (4). For each place v of K , we have a map $A_0(X)_0 \rightarrow A_0(X_v)_0$, where $X_v = X \times K_v$ and K_v is the completion of K at the place v . We also have a map $j_v : A_0(X_v)_0 \rightarrow (\mathbf{Z}/2\mathbf{Z})^3$ (restoring symmetry) whose image lies in the subgroup $b_1 + b_2 + b_3 = 0$. If $d \in K_v^{\times 2}$, then $A_0(X_v)_0 = \{0\}$ [1, Prop. 4.7], as X_v is then K_v -birational to \mathbf{P}_2 . Assume that $d \notin K_v^{\times 2}$.

If v is a real place, the results of [2] allow us to compute j_v . Assume that v does not lie above ∞ .

Suppose that the extension $K_v(\sqrt{d})$ is unramified. If v lies above an odd prime, the results of [6, Prop. 4.7] (cf. Prop. 1; see also [7, n° 4]) allow us to compute j_v . If v lies above 2, then one applies [7, n° 4].

Suppose now that the extension $K_v(\sqrt{d})$ is ramified. If v lies above an odd prime, one can use [7, Prop. 2] (cf. Prop. 2). Finally, if v divides 2, one applies Prop. 3 to compute j_v .

By the local-to-global principle as conjectured in [4] and proved in [5], the group $A_0(X)_0$ is the kernel of the map $\oplus_v j_v$ from $\oplus_v A_0(X_v)_0$ into $(\mathbf{Z}/2\mathbf{Z})^3$.

BIBLIOGRAPHIC REFERENCES

- [1] COLLIOT-THÉLÈNE (J-L.) and CORAY (D. F.). — *L'équivalence rationnelle sur les points fermés des surfaces rationnelles fibrées en coniques*, Compositio Math., **39** (3), 1979, pp. 301–332
- [2] COLLIOT-THÉLÈNE (J-L.) and ISCHEBECK (F.). — *L'équivalence rationnelle sur les cycles de dimension zéro des variétés algébriques réelles*, Comptes-rendus de l'Acad. des sci. **292** (1981), 723–725.
- [3] COLLIOT-THÉLÈNE (J-L.) and SANSUC (J-J.). — *La descente sur les variétés rationnelles*, dans Journées de Géométrie algébrique d'Angers, Alpen aan den Rijn : Sijthoff & Noordhoff, 1980.
- [4] COLLIOT-THÉLÈNE (J-L.) and SANSUC (J-J.). — *On the Chow groups of certain rational surfaces : a sequel to a paper of S. Bloch*, Duke Math. J., **48** (2), 1981, pp. 421–447
- [5] COLLIOT-THÉLÈNE (J-L.), SANSUC (J-J.) and SWINNERTON-DYER (H. P. F.). — *Intersections of two quadrics and Châtelet surfaces. I, II*. J. Reine Angew. Math. **373** (1987), pp. 37–107; *ibid.* **374** (1987), pp. 72–168.

- [6] CORAY (D. F.) and TSFASMAN (M. A.). — *Arithmetic on singular del Pezzo surfaces*, Proc. London Math. Soc. (3) **57** (1), 1988, pp. 25–87.
- [7] DALAWAT (C. S.). — *Le groupe de Chow d’une surface de Châtelet sur un corps local*, Indagationes math., **11** (2), 2000, pp. 171–185.
- [8] SWINNERTON-DYER (H. P. F.). — *Two descent from Fermat to now*, Mathematisches Institut Universität Göttingen, Summer Term 2004, pp. 95–102, Universitätsdrucke Göttingen, Göttingen, 2004.
- [9] SANSUC (J.-J.). — *À propos d’une conjecture arithmétique sur le groupe de Chow d’une surface rationnelle*, Séminaire de Théorie des nombres de Bordeaux, Exposé 33, 1982.

Chandan Singh Dalawat
 Harish-Chandra Research Institute
 Chhatnag Road, Jhansi
 ALLAHABAD 211 019, India
 dalawat@mri.ernet.in